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An Extension of Pythagorean and Isosceles Orthogonality and a Characterization of Inner-Product Spaces

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A new orthogonality relation in normed linear spaces which generalizes pythagorean orthogonality and isosceles orthogonality is defined, and it is shown that the new orthogonality is homogeneous (additive) if and only if the space is a real inner-product space.

0. INTRODUCTION

In this paper a new form of orthogonality is defined in normed linear spaces, and a new characterization of real inner-product spaces is given in terms of the new orthogonality relation.

Let X denote a normed linear space over the reals with norm $\|\cdot\|$, and let $\alpha \neq 1$ be a fixed real number. We define the concept of α-orthogonality as follows. For $x, y \in X$, we say x is *α-orthogonal* to y, denoted $(x \perp y)(\alpha)$, if and only if $(1+\alpha^2)\|x-y\|^2 = \|x-\alpha y\|^2 + \|y-\alpha x\|^2$. The geometrical motivation for this definition is clear. It is also clear that α-orthogonality is symmetric, i.e., if $x, y \in X$, $(x \perp y)(\alpha)$ if and only if $(y \perp x)(\alpha)$. Furthermore, pythagorean orthogonality and isosceles orthogonality (see James [1]) are special cases of α-orthogonality for $\alpha = 0$ and $\alpha = -1$, respectively. It is easy to see that if $(x \perp x)(\alpha)$, then $x = 0$ and that if $(X, \|\cdot\|)$ is a real inner-product space with inner-product $(\cdot \mid \cdot)$, $(x \perp y)(\alpha)$ if and only if $(x \mid y) = 0$. Furthermore, a simple example, the 2-dimensional

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Minkowski space with $\|(x_1, x_2)\| = |x_1| + |x_2|$, shows that for $\alpha \neq 0, -1$, α-orthogonality is distinct from pythagorean orthogonality and isosceles orthogonality.

In Section 1, further properties of α-orthogonality are derived and in Section 2 a characterization of real inner-product spaces is obtained in terms of homogeneity (additivity) of the relation (see [1]). The main result is the following:

THEOREM. *If α-orthogonality is homogeneous or additive, then the space $(X, \|\cdot\|)$ is a real inner-product space.*

1. PROPERTIES OF α-ORTHOGONALITY

If $\alpha \neq 0, 1$, then it is easily seen from the definition of α-orthogonality that $(x \perp y)(\alpha)$ if and only if $(x \perp y)(1/\alpha)$. Therefore, there is no loss of generality if we assume that $\alpha \in [-1, 1)$. Also, for any real k, Lemma 4.4 of [1] implies that

$$\lim_{t \to \infty} [\|(t+k)x + y\| - \|tx + y\|] = k\|x\|. \tag{1}$$

THEOREM 1. *If $\alpha \neq 1$, then for any x and y there is a real number a such that $(x \perp ax + y)(\alpha)$.*

Proof. Since the statement clearly is true if $x = 0$, we will assume that $x \neq 0$. Define

$$g(t) = \|x - \alpha(tx + y)\|^2 + \|tx + y - \alpha x\|^2 - (1 + \alpha^2)\|x - (tx + y)\|^2.$$

We will show that $\lim_{t \to \infty} g(t) = \infty$ and $\lim_{t \to -\infty} g(t) = -\infty$, so the continuity of $g(t)$ will imply the existence of a real number a for which $g(a) = 0$ and hence $(x \perp ax + y)(\alpha)$. Since

$$g(t) = \alpha^2 \left[\left\| \left[t + \left(1 - \frac{1}{\alpha}\right) \right] x + (y - x) \right\|^2 - \|tx + (y - x)\|^2 \right] + [\|[t + (1 - \alpha)]x + (y - x)\|^2 - \|tx + (y - x)\|^2],$$

it follows from (1) that there are functions ε_1, ε_2, ε_3, and ε_4, which approach 0 as $t \to \infty$ and for which

$$g(t) = [\varepsilon_1 + \alpha(\alpha - 1)\|x\|] \left[\left\| \left[t + \left(1 - \frac{1}{\alpha}\right) \right] x + (y - x) \right\| + \|tx + (y - x)\| \right] + [\varepsilon_2 + (1 - \alpha)\|x\|][\|[t + (1 - \alpha)]x + (y - x)\| + \|tx + (y - x)\|]$$

$$= [\varepsilon_1 + \alpha(\alpha-1)\|x\|]\left[2\|tx+(y-x)\| + \left(1-\frac{1}{\alpha}\right)\|x\| + \varepsilon_3\right]$$

$$+ [\varepsilon_2 + (1-\alpha)\|x\|][2\|tx+(y-x)\| + (1-\alpha)\|x\| + \varepsilon_4]$$

$$= 2[\varepsilon_1 + \varepsilon_2 + (\alpha-1)^2\|x\|]\|tx+(y-x)\|$$

$$+ [\varepsilon_1 + \alpha(\alpha-1)\|x\|]\left[\left(1-\frac{1}{\alpha}\right)\|x\| + \varepsilon_3\right]$$

$$+ [\varepsilon_2 + (1-\alpha)\|x\|][(1-\alpha)\|x\| + \varepsilon_4].$$

Since $\alpha \neq 1$, this approaches ∞ as $t \to \infty$. The proof that $\lim_{t\to-\infty} g(t) = -\infty$ is similar. This completes the proof of the theorem.

2. The Characterization Theorem

In this section, we make additional assumptions about our α-orthogonality relation and derive characterizations of inner-product spaces. As before, let $(X, \|\cdot\|)$ denote a normed linear space over the reals. Following James [1], we say that α-orthogonality is *homogeneous* if for all $x, y \in X$, a, b real numbers, $(x \perp y)(\alpha)$ implies $(ax \perp by)(\alpha)$, and that α-orthogonality is *additive* if for all $x, y, z \in X$, if $(x \perp y)(\alpha)$ and $(x \perp z)(\alpha)$, then $(x \perp y+z)(\alpha)$. We will show that if α-orthogonality is homogeneous or additive, then the space $(X, \|\cdot\|)$ is a real inner-product space. Since the result is known if $\alpha = -1$ (isosceles orthogonality) (James [1]), by the comment made at the beginning of Section 1, we assume in the following that $|\alpha| < 1$.

Theorem 2. *If α-orthogonality is homogeneous, then $(x \perp y)(\alpha)$ if and only if $\|x-y\|^2 = \|x\|^2 + \|y\|^2$, i.e., α-orthogonality is equivalent to pythagorean orthogonality.*

Proof. Since the statement is obvious if $x = 0$ or $y = 0$ we will assume in the following that x and y are nonzero. Also, as noted previously, it suffices to consider $|\alpha| < 1$.

(1) Let $(x \perp y)(\alpha)$. We begin by establishing the following statement:

$$P(n): \qquad (1+\alpha^{2^n})\|x-y\|^2 = \|x-\alpha^{2^{n-1}}y\|^2 + \|y-\alpha^{2^{n-1}}x\|^2.$$

$P(1)$ is equivalent to $(x \perp y)(\alpha)$. If $P(n)$ is true and $(x \perp y)(\alpha)$, then letting $\beta = \alpha^{2^{n-1}}$ and using homogeneity, we may replace x by βx in $P(n)$ and then replace y by βy in $P(n)$ to obtain

$$(1+\beta^2)\|\beta x - y\|^2 = \beta^2\|x-y\|^2 + \|y-\beta^2 x\|^2$$

and

$$(1+\beta^2)\,\|x-\beta y\|^2 = \|x-\beta^2 y\|^2 + \beta^2\,\|x-y\|^2.$$

Now add corresponding members of these equalities and use $P(n)$ to obtain

$$(1+\beta^2)^2\,\|x-y\|^2 = 2\beta^2\,\|x-y\|^2 + \|x-\beta^2 y\|^2 + \|y-\beta^2 x\|^2$$

and

$$(1+\beta^4)\,\|x-y\|^2 = \|x-\beta^2 y\|^2 + \|y-\beta^2 x\|^2$$

and the assertion follows by mathematical induction.

Since $|\alpha| < 1$, we get the desired result by taking the limit as $n \to \infty$ in $P(n)$.

(2) Conversely, assume that x and y satisfy the condition

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2. \tag{2}$$

By Theorem 1, there is a real number a such that $(x \perp ax+y)(\alpha)$. By homogeneity of α-orthogonality, we also have $(\beta x \perp ax+y)(\alpha)$ for any real β, and hence by part 1 of this proof,

$$\|(\beta-a)x - y\|^2 = \beta^2\,\|x\|^2 + \|ax+y\|^2 \tag{3}$$

for any real β. Applying (3) for $\beta = a$ and for $\beta = a+1$ and then combining these results with (2), we obtain

$$\|x\|^2 = (2a+1)\,\|x\|^2$$

and hence, $a = 0$. This completes the proof of the theorem.

By using Theorem 5.2 of [1] and a proof similar to that used in Theorem 5.3 of [1], the main result now follows easily.

THEOREM 3. *If α-orthogonality is homogeneous or additive, then the space $(X, \|\cdot\|)$ is a real inner-product space.*

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REFERENCE

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